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SOME ASPECTS OF THE OPTIMAL CONTROL OF NON-LINEAR DESCRIPTOR SYSTEMS†

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The modification of the algorithms of the calculus of variations and Pontryagin's maximum principle required for them to be applicable to non-linear descriptor control systems is demonstrated. The classical calculus of variations is still applicable in optimization without constraints on the control, but when such constraints are imposed, the application of Pontryagin's maximum principle in its standard or extended form requires a distinction to be made between proper and non-proper descriptor systems. In the non-proper case, the solution depends on higher-order time derivatives of the control inputs. For the correct description of the problem and for Pontryagin's maximum principle to be applicable additional phase variables and corresponding integrator chains have to be introduced. The optimal control thus obtained becomes dynamic. To simplify the notation, the index of the algebraic equations of the constraints is assumed to be uniform. In principle, the results remain valid for a non-uniform index also, i.e. when different constraint equations have different indices and different components of the control occur with different maximum orders of the time derivatives. The results are somewhat complicated, particularly in the case of constrained optimization of non-proper systems. © 2002 Elsevier Science Ltd. All rights reserved.

Algorithms for constructing the optimal control in the calculus of variations and in Pontryagin's maximum principle will be extended here to descriptor control systems (DS), described by systems of ordinary differential and algebraic equations. An important role is played in that context by the concepts of "proper" and "non-proper" DS, namely, those whose behaviour depends only on the control inputs and those which also involve higher-order time derivatives of the control inputs. In the case of non-proper systems, the formulation of the optimal control problem must be modified.

Discussion of DS originated in 1977 with the fundamental paper [1]. Since that time, considerable progress has been made in investigating such systems (see surveys, [2, 3] for linear DS, the first results for non-linear DS in [4–6], the first attempts to construct optimal controls in [7, 8], and an analysis of the linear-quadratic optimal regulator for DS in [9], as well as the latest theoretical publications on optimal control problems with phase constraints [10–12]. In most of the publications, however, considerable restrictions on the type of control problem have been introduced to obtain the necessary optimum conditions. These restrictions are sometimes not realized in practical, real-life problems. A correct solution is only possible if allowance is made for differences between proper and non-proper DS, as we have in [13, 14] when constructing linear-quadratic optimal controls for linear DS. This paper will discuss the special features of the construction of the optimal control for non-linear DS, in relation to proper and non-proper systems behaviour.

1. FORMULATION OF THE PROBLEM AND DIFFERENT REPRESENTATIONS OF THE SYSTEM

A controlled time-invariant finite-dimensional DS is described by the differential-algebraic equations

$$\dot{\mathbf{x}}_1 = \mathbf{f}_1(\mathbf{x}_1, \mathbf{x}_2, \mathbf{u}) \tag{1.1}$$

$$\mathbf{0} = \mathbf{f}_2(\mathbf{x}_1, \mathbf{x}_2, \mathbf{u}) \tag{1.2}$$

where $\mathbf{x}_i$, $\mathbf{f}_i$ $(i = 1, 2)$ are n_i-dimensional vectors $(n_1 + n_2 = n)$ and $\mathbf{u} \in U$ is the r-dimensional control input vector.

The optimal control problem for a non-linear DS is to construct a control that will minimize the functional (performance criterion)

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$$J = \int_0^T \mathbf{f}_0(\mathbf{x}_1, \mathbf{x}_2, \mathbf{u})dt \Rightarrow \min \tag{1.3}$$

where $\mathbf{u}(t)$ belongs to a set of bounded or unbounded control functions. In addition, the construction of an optimal control requires the introduction of boundary conditions, whether these are given as geometric relations or dynamic boundary conditions established during the actual solution of the optimization problem. In either case they must be compatible with algebraic equations (1.2). Since the use of boundary conditions for constructing optimal controls in a DS is no different from the usual rules for constructing optimal controls, boundary conditions will not be explicitly considered in what follows.

Only general principles governing the correct formulation of optimal control problems for non-linear DS will be considered; to simplify the notation, moreover, it will be assumed that algebraic equations (1.2) have a uniform index (i.e. all the equations have the same index).

For accurate investigation of DS, it is useful to bear in mind the different possible representations of a dynamical system (1.1)–(1.3), obtained either by constructing a differential equation for $\mathbf{x}_2$ by repeated differentiation of algebraic equations (1.2), or by eliminating redundant coordinates and deriving a differential equation in phase space. In either case, a key role is played by the index of the system, that is, the number k of differentiations of algebraic equations (1.2) needed to derive the underlying set of ordinary differential equations. Let us assume that $\mathbf{f}_1$ and $\mathbf{f}_2$ are, respectively, $k-1$ and k times continuously differentiable functions, and consider, together with $\mathbf{f}_2$, its total derivatives with respect to time $\dot{\mathbf{f}}_2$, $\ddot{\mathbf{f}}_2$, ..., $\mathbf{f}_2^{(k-1)}$, $\mathbf{f}_2^{(k)}$ along trajectories of system (1.1). It is assumed that in the case of a uniform index k the functions $\mathbf{f}_2, \dot{\mathbf{f}}_2, \ldots, \mathbf{f}_2^{(k-2)}$ depend on $\mathbf{x}_1$ but not on $\mathbf{x}_2$, and the first function to depend on $\mathbf{x}_2$ is $\mathbf{f}_2^{(k-1)} = 0$, but then in such a way that its Jacobian is non-singular and the equation $\mathbf{f}_2^{(k)} = 0$ is solvable for $\dot{\mathbf{x}}_2$.

We introduce the notation

$$L(\cdot) = L_{f_1}(\cdot) + L_{\dot{u}}(\cdot) + \frac{\Delta}{\Delta t}(\cdot) \tag{1.4}$$

$$L_{f_1}(\cdot) = \frac{\partial(\cdot)}{\partial \mathbf{x}_1^T}\mathbf{f}_1, \quad L_{u^{(j)}}(\cdot) = \frac{\partial(\cdot)}{\partial \mathbf{u}^T}\mathbf{u}^{(j)}, \quad \frac{\Delta}{\Delta t} L_{u^{(j)}}(\cdot) = L_{u^{(j+1)}}(\cdot) \tag{1.5}$$

where operator (1.4) is defined in terms of operators (1.5), the first two of which are Lie derivatives, while the last acts only on $L_{u(j)}$ by time differentiation of the control input functions.

In this notation, we have

$$\mathbf{f}_2^{(j)} \equiv L^j(\mathbf{f}_2) = \mathbf{0}, \quad j = 0, \ldots, k-1 \tag{1.6}$$

$$\dot{\mathbf{x}}_2 = -\left(\frac{\partial}{\partial \mathbf{x}_2^T} L_{f_1}^{k-1}(\mathbf{f}_2)\right)^{-1}, \quad L^k(\mathbf{f}_2) = \bar{\mathbf{f}}_2(\mathbf{x}_1, \mathbf{x}_2, \mathbf{u}, \ldots, \mathbf{u}^{(s)}) \tag{1.7}$$

Here relations (1.6) are first integrals of differential equations (1.7), and depend in the general case on the control inputs $\mathbf{u}$ and their time derivatives $\mathbf{u}, \ldots, \mathbf{u}^{(s)}$, $0 \leqslant s \leqslant k$. If $\mathbf{u}$ first occurs explicitly in $L^p(\mathbf{f}_2) = \mathbf{0}$, then $s = k - p$.

Thus, the DS (1.1), (1.2) may be represented by differential equations (1.1) and (1.7) on the invariant manifold defined by the first integrals (1.6).

In principle, one can use invariants (1.6) to eliminate kn_2 redundant variables and reduce the problem to a system of $n_1 - (k-1)n_2$ ordinary differential equations in phase space. But we shall not follow that path.

Both types of representation for DS require, in addition, appropriate initial or boundary conditions satisfying the first integrals (1.6):

$$L^j(\mathbf{f}_2)|_{t=0} = \mathbf{0}, \quad L^j(\mathbf{f}_2)|_{t=T} = \mathbf{0}, \quad j = 0, \ldots, k-1 \tag{1.8}$$

2. PROPER AND NON-PROPER DS

The different representations of DS in the preceding section indicate that the behaviour of a DS may depend not only on the control input $\mathbf{u}$ but also on its time derivatives $\dot{\mathbf{u}}, \ddot{\mathbf{u}}, \ldots, \mathbf{u}^{(s)}$. Irrespective of whether the differential-algebraic description (1.1), (1.2) includes explicit dependence only on the input u, hidden effects depending on the time derivatives $\dot{\mathbf{u}}$, $\ddot{\mathbf{u}}$, ..., $\mathbf{u}^{(s)}$ may appear, as is obvious from representation (1.1), (1.6) and (1.7). Any method for constructing controls must therefore make allowance for this fact, which does not arise in the usual phase-space treatment. In discrete time systems the unusual situation just outlined is cleared up [3] by introducing the concept of "causality," which unfortunately has no analogue in continuous-time systems.

For continuous-time systems, we introduce the concept of "properness," corresponding to the definition of the frequency domain for linear systems. The DS (1.1), (1.2) is said to be *proper* if the solution $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$ depends only on $\mathbf{u}(t)$ (and possibly on integrals of $\mathbf{u}(t)$) but not on $\dot{\mathbf{u}}(t)$, $\ddot{\mathbf{u}}(t)$, ..., $\mathbf{u}^{(s-1)}(t)$. Otherwise, the system is said to be *non-proper*.

This definition does not necessarily involve $\mathbf{u}^{(s)}(t)$, as might be expected on the basis of (1.7), because $L^{k-1}(\mathbf{f}_2) = \mathbf{0}$ is a first integral of system (1.7) which depends only on $\mathbf{u}(t)$, $\dot{\mathbf{u}}(t)$, $\ddot{\mathbf{u}}(t)$, ..., $\mathbf{u}^{(s-1)}(t)$. Note that a DS of index $k = 1$ is always proper. Non-proper systems crop up only when $k \geqslant 2$. In addition, it follows from (1.6) and (1.7) and the definition of "properness" that a DS of index k is proper if and only if

$$\frac{\partial}{\partial \mathbf{u}^T}(L^j_{f_1}(\mathbf{f}_2)) = \mathbf{0}, \quad j = 0, \ldots, k-2$$

Then the first integrals (1.6) become

$$L^j(\mathbf{f}_2) \equiv L^j_{f_1}(\mathbf{f}_2) \equiv \mathbf{f}_{2,j}(\mathbf{x}_1) = \mathbf{0}, \quad j = 0, \ldots, k-2 \tag{2.1}$$

$$L^{k-1}(\mathbf{f}_2) \equiv L^{k-1}_{f_1}(\mathbf{f}_2) \equiv \mathbf{f}_{2,k-1}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{u}) = \mathbf{0} \tag{2.2}$$

They do not depend on time derivatives of the control input, and the condition $s = 1$ is satisfied.

A stronger condition is also considered: the DS (1.1) is *strictly proper* if it is proper and, additionally, $\mathbf{f}_{2,k-1}$ depends on $\mathbf{x}_1$ but not on $\mathbf{u}$, so that $s = 0$ and function (2.2) becomes

$$\mathbf{f}_{2,k-1}(\mathbf{x}_1, \mathbf{x}_2) = \mathbf{0} \tag{2.3}$$

Properness plays an important role for Pontryagin's maximum principle to be applicable in its standard form.

As an example of a proper DS, we will describe a typical simple control mechanism consisting of two masses attached at the ends of springs (linear oscillators) and connected by a solid rod of variable length, which is regarded as the control input (see Fig. 1).

Considering the variables z_1 and z_2 as the displacements of the masses from their static equilibrium positions z_{10} and z_{20}, we can write Lagrange's equations of the second kind in the form

$$m_1\ddot{z}_1 + c_1 z_1 = \lambda, \quad m_2\ddot{z}_2 + c_2 z_2 = -\lambda, \quad z_1 - z_2 + u = 0 \tag{2.4}$$

Equations (2.4) take the descriptor form (1.1), (1.2) if we introduce the notation

$$\begin{aligned} &\mathbf{x}_1 = [z_1, z_2, \dot{z}_1, \dot{z}_2]^T, \quad \mathbf{x}_2 = [\lambda] \\ &\mathbf{f}_1 = \left[\dot{z}_1, \dot{z}_2, -\frac{c_1}{m_1}z_1 + \frac{\lambda}{m_1}, -\frac{c_2}{m_2}z_2 - \frac{\lambda}{m_2}\right]^T \\ &\mathbf{f}_2 = [z_1 - z_2 + u] \end{aligned} \tag{2.5}$$

By the change variables

$$\bar{x}_1 = \frac{2}{m_1 + m_2}(m_1 z_1 + m_2 z_2), \quad \bar{x}_2 = \frac{2}{m_1 + m_2}(m_1\dot{z}_1 + m_2\dot{z}_2)$$

$$\bar{x}_3 = -\frac{c_1}{m_1}z_1 + \frac{c_2}{m_2}z_2 + \frac{m_1 + m_2}{m_1 m_2}\lambda, \quad \bar{x}_4 = \dot{z}_1 - \dot{z}_2, \quad \bar{x}_5 = z_1 - z_2$$

we transform the system to Weierstrass–Kronecker canonical form [3], which leads to the solution of Eqs (2.4)

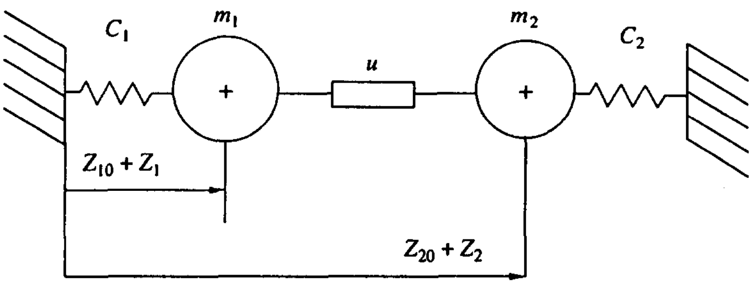

Fig. 1

$$z_1(t) = \frac{1}{2}\bar{x}_1(t) - \frac{m_2}{m_1 + m_2}u(t), \quad z_2(t) = \frac{1}{2}\bar{x}_1(t) + \frac{m_1}{m_1 + m_2}u(t) \tag{2.6}$$

$$\dot{z}_1(t) = \frac{1}{2}\dot{\bar{x}}_1(t) - \frac{m_2}{m_1 + m_2}\dot{u}(t), \quad \dot{z}_2(t) = \frac{1}{2}\dot{\bar{x}}_1(t) + \frac{m_1}{m_1 + m_2}\dot{u}(t) \tag{2.7}$$

$$\lambda(t) = \frac{c_1 m_2 - c_2 m_1}{2(m_1 + m_2)}\bar{x}_1(t) - \frac{c_1 m_2^2 + c_2 m_1^2}{(m_1 + m_2)^2}u(t) - \frac{m_1 m_2}{m_1 + m_2}\ddot{u}(t) \tag{2.8}$$

where $\bar{x}_1(t)$ satisfies the ordinary differential equation

$$\ddot{\bar{x}}_1(t) + \frac{c_1 + c_2}{m_1 + m_2}\bar{x}_1(t) = 2\frac{c_1 m_2 - c_2 m_1}{(m_1 + m_2)^2}u(t)$$

Obviously, the solution depends on $\dot{u}(t)$ (2.7) and on $\ddot{u}(t)$ (2.8). From a mechanical standpoint, this result is not surprising. If displacements by virtue of the last of equations (2.4) depend on $u(t)$, then velocities (2.7) will depend on $\dot{u}(t)$ and the reaction of constraint (2.8) will depend on the acceleration $\ddot{u}(t)$. Nevertheless, system (2.4) is non-proper.

In general, it is not difficult to show that Lagrangian systems are proper if and only if the control input vector appears in differential equation (1.1) but not in algebraic equations (1.2), that is, in the case of ordinary control by force or torque. But if the control input $\mathbf{u}(t)$, via algebraic equations (1.2), constrains the displacement of the system, as in the example just considered, the behaviour of the system turns out to be non-proper.

3. OPTIMAL CONTROL (GENERAL PRINCIPLES)

To derive the necessary conditions for the optimal control in a DS with performance criterion (1.3), using the well-known methods of the calculus of variations or Pontryagin's maximum principle [15–17], the most convenient description for a DS is a system consisting of differential equations (1.1) and (1.7) and additional boundary conditions (1.8). In the case of a non-proper system, correct treatment of the time derivatives $\dot{\mathbf{u}}(t)$, $\ddot{\mathbf{u}}(t)$, ..., $\mathbf{u}^{(s)}(t)$. is necessary. This problem is easily solved by introducing the extended set of variables

$$\boldsymbol{\xi}_1 = \mathbf{u}, \ \boldsymbol{\xi}_2 = \dot{\mathbf{u}}, \ldots, \ \boldsymbol{\xi}_s = \mathbf{u}^{(s-1)}, \ \mathbf{v} = \mathbf{u}^{(s)} \tag{3.1}$$

which define a multi-dimensional integrator chain

$$\dot{\boldsymbol{\xi}}_1 = \boldsymbol{\xi}_2, \ \dot{\boldsymbol{\xi}}_2 = \boldsymbol{\xi}_3, \ldots, \ \dot{\boldsymbol{\xi}}_{s-1} = \boldsymbol{\xi}_s, \ \dot{\boldsymbol{\xi}}_s = \mathbf{v} \tag{3.2}$$

The idea of introducing integrator chains (3.1), (3.2) for a correct description of the influence of the control input has been used to construct a linear-quadratic optimal control of a non-proper linear DS. While the usual equations (see Eqs (4.3) and (4.4) below) remain valid in this case, the correct solution of a two-point boundary-value problem is greatly simplified by introducing integrator chain (3.1), (3.2). The so-called Riccati approach may also be applied correctly only by introducing the extension (3.1), (3.2). For non-proper systems, the amplification matrix for dynamical feedback is determined by a Riccati matrix equation for the extended dynamical system. The details of this approach are set out in [13, 14].

With additional variables, the optimal control problem becomes the following. It is required to minimize the performance criterion (1.3)

$$J = \int_0^T f_0(\mathbf{x}_1, \mathbf{x}_2, \boldsymbol{\xi}_1)dt \Rightarrow \min$$

in the presence of differential constraints corresponding to the extended dynamical system (1.1), (1.6), (3.2)

$$\dot{\mathbf{x}}_e = [\mathbf{f}_1(\mathbf{x}_1, \mathbf{x}_2, \boldsymbol{\xi}_1),\ \ \bar{\mathbf{f}}_2(\mathbf{x}_1, \mathbf{x}_2, \boldsymbol{\xi}_1, \ldots, \mathbf{v}),\ \ \boldsymbol{\xi}_2, \ldots, \boldsymbol{\xi}_s, \mathbf{v}]^T \tag{3.3}$$

where the extended phase vector is defined as

$$\mathbf{x}_e = [\mathbf{x}_1, \mathbf{x}_2, \boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_s]^T \tag{3.4}$$

and the vector $\mathbf{v}(t)$ is a new (fictitious) control input. The original constraint on the control, $\mathbf{u} \in U$, is now replaced by a phase constraint $\xi_1 \in U$. The given geometrical boundary conditions must be supplemented by conditions (1.8).

Introducing Lagrange multipliers $\boldsymbol{\lambda}_1, \bar{\boldsymbol{\lambda}}_2, \boldsymbol{\psi}_1, \boldsymbol{\psi}_2, \ldots, \boldsymbol{\psi}$ into the extended formulation of the problem, we obtain the extended performance criterion

$$J_e = \int_0^T [f_0 + \boldsymbol{\lambda}_1^T(\dot{\mathbf{x}}_1 - \mathbf{f}_1) + \bar{\boldsymbol{\lambda}}_2^T(\dot{\mathbf{x}}_2 - \bar{\mathbf{f}}_2) +$$

$$+ \boldsymbol{\psi}_1^T(\dot{\boldsymbol{\xi}}_1 - \boldsymbol{\xi}_2) + \ldots + \boldsymbol{\psi}_{s-1}^T(\dot{\boldsymbol{\xi}}_{s-1} - \boldsymbol{\xi}_s) + \boldsymbol{\psi}_s^T(\dot{\boldsymbol{\xi}}_s - \mathbf{v})]dt \Rightarrow \min \tag{3.5}$$

and the Hamiltonian

$$H = \boldsymbol{\lambda}_1^T\mathbf{f}_1 + \bar{\boldsymbol{\lambda}}_2^T\bar{\mathbf{f}}_2 - f_0 + \boldsymbol{\psi}_1^T\boldsymbol{\xi}_2 + \boldsymbol{\psi}_2^T\boldsymbol{\xi}_3 + \ldots + \boldsymbol{\psi}_{s-1}^T\boldsymbol{\xi}_s + \boldsymbol{\psi}_s^T\mathbf{v}$$

An optimal control for a non-proper DS of general form, in the unconstrained case ($U = R^r$), is now constructed using the classical calculus of variations; if there are constraints ($U \subset R^r$), Pontryagin's maximum principle is used.

This general approach may be simplified for unconstrained optimization problems and for proper systems.

4. THE OPTIMIZATION PROBLEM WHEN THERE ARE NO CONSTRAINTS ON THE CONTROL (THE CALCULUS OF VARIATIONS)

Let us consider performance criterion (3.5) in greater detail, in particular, its third term

$$J_2 = \int_0^T \bar{\boldsymbol{\lambda}}_2^T(\dot{\mathbf{x}}_2 - \bar{\mathbf{f}}_2)dt = \int_0^T \bar{\boldsymbol{\lambda}}_2^T\mathbf{F}^{-1}(\mathbf{F}\dot{\mathbf{x}}_2 + L^k(\mathbf{f}_2))dt \quad \left(\mathbf{F} = \frac{\partial}{\partial \mathbf{x}_2^T} L_{f_1}^{k-1}(\mathbf{f}_2)\right)$$

Introducing $\boldsymbol{\eta}^T = \bar{\boldsymbol{\lambda}}_2^T\mathbf{F}^{-1}$ after integrating by parts taking Eqs (1.6)–(1.8) into consideration, we obtain

$$J_2 = (-1)^k \int_0^T \boldsymbol{\eta}^{(k)T}\mathbf{f}_2 dt$$

Putting $\lambda_2 = -(-1)^k\boldsymbol{\eta}^{(k)}$, we reduce the performance criterion to the form

$$J_e = \int_0^T [f_0 + \boldsymbol{\lambda}_1^T(\dot{\mathbf{x}}_1 - \mathbf{f}_1) - \boldsymbol{\lambda}_2^T\mathbf{f}_2 + \boldsymbol{\psi}_1^T(\dot{\boldsymbol{\xi}}_1 - \boldsymbol{\xi}_2) + \ldots + \boldsymbol{\psi}_s^T(\dot{\boldsymbol{\xi}}_s - \mathbf{v})]dt \Rightarrow \min \tag{4.1}$$

First writing out Euler's equation for $\mathbf{v}$, we obtain $\psi_s = 0$. Euler's equations for $\xi_2, \ldots, \xi_s$ yield equalities $\psi_i = \mathbf{0}$ $(i = 1, \ldots, s-1)$. Using the reduced Hamiltonian

$$H_r = H_r(\mathbf{x}_1, \mathbf{x}_2, \mathbf{u}, \boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2) = \boldsymbol{\lambda}_1^T \mathbf{f}_1 + \boldsymbol{\lambda}_2^T \mathbf{f}_2 - f_0 \tag{4.2}$$

we can express the remaining Euler's equations in the form

$$\dot{\boldsymbol{\lambda}}_1 = -\frac{\partial H_r}{\partial \mathbf{x}_1}, \quad \mathbf{0} = -\frac{\partial H_r}{\partial \mathbf{x}_2} \tag{4.3}$$

$$\mathbf{0} = -\frac{\partial H_r}{\partial \mathbf{u}} \tag{4.4}$$

The conjugate variables $\boldsymbol{\lambda}_1$ and $\boldsymbol{\lambda}_2$ satisfy the differential-algebraic equations (4.3), and the optimal control is determined from Eq. (4.4); it generally depends on $\mathbf{x}_1$, $\mathbf{x}_2$, $\boldsymbol{\lambda}_1$, $\boldsymbol{\lambda}_2$.

The boundary conditions for the conjugate variables are defined according to the usual rules and will not be considered here explicitly.

The boundary conditions for the conjugate variables are defined according to the usual rules and will not be considered here explicitly.

The procedure for solving unconstrained optimization problems is to solve a two-point boundary-value problem for two subsystems of differential-algebraic equations: subsystem (1.1), (1.2) and subsystem (4.3), in which the control is determined from Eq. (4.4). Thus an unconstrained optimization problem is solved using the standard approach of the calculus of variations, irrespective of whether the dynamical system (1.1), (1.2) is proper or not.

5. OPTIMIZATION PROCEDURE FOR A PROPER DS WITH CONSTRAINTS ON THE CONTROL

Proper DS are characterized by the invariants (2.1) and (2.2) with $s = 1$. Consequently, the extended set of variables (3.1) includes only

$$\boldsymbol{\xi} = \mathbf{u}, \quad \mathbf{v} = \dot{\mathbf{u}} : \dot{\boldsymbol{\xi}} = \mathbf{v} \tag{5.1}$$

Then performance criterion (4.1) (transformed as a result of integration by parts) becomes

$$J_e = \int_0^T [f_0 + \boldsymbol{\lambda}_1^T(\dot{\mathbf{x}}_1 - \mathbf{f}_1) - \boldsymbol{\lambda}_2^T \mathbf{f}_2 + \boldsymbol{\psi}^T(\dot{\boldsymbol{\xi}} - \mathbf{v})]dt \Rightarrow \min$$

leading to a Hamiltonian $H_p = \boldsymbol{\lambda}_1^T \mathbf{f}_1 + \boldsymbol{\lambda}_2^T \mathbf{f}_2 + \boldsymbol{\psi}^T \mathbf{v} - f_0$ and to the necessary conditions of Pontryagin's maximum principle

$$\dot{\boldsymbol{\lambda}}_1 = -\frac{\partial H_p}{\partial \mathbf{x}_1}, \quad \mathbf{0} = -\frac{\partial H_p}{\partial \mathbf{x}_2}, \quad \dot{\boldsymbol{\psi}} = -\frac{\partial H_p}{\partial \boldsymbol{\xi}}, \quad H_{p\max} = \max_v H_p : \boldsymbol{\psi}^T \mathbf{v} \Rightarrow \max_v$$

If in addition the DS is strictly proper, so that $s = 0$, and condition (2.3) is satisfied, then extension (5.1) is not needed at all, and we again obtain the previously presented Hamiltonian H_r (4.2), yielding necessary conditions

$$\dot{\boldsymbol{\lambda}}_1 = -\frac{\partial H_r}{\partial \mathbf{x}_1}, \quad \mathbf{0} = -\frac{\partial H_r}{\partial \mathbf{x}_2}, \quad H_{r\max} = \max_{u \in U} H_r$$

Consequently, the constrained optimization problem for a strictly proper DS is solved by the usual means, employing Pontryagin's maximum principle.

6. OPTIMIZATION PROCEDURE FOR A NON-PROPER DS WITH CONSTRAINTS ON THE CONTROL

For a non-proper DS (1.1), (1.2) one can apply only the general approach mentioned in Section 3. For extended system (3.3) with Hamiltonian H_{np}, associated with performance criterion (4.1)

$$H_{np} = \boldsymbol{\lambda}_1^T \mathbf{f}_1 + \boldsymbol{\lambda}_2^T \mathbf{f}_2 + \boldsymbol{\psi}_1^T \boldsymbol{\xi}_2 + \ldots + \boldsymbol{\psi}_s^T \mathbf{v} - f_0$$

the necessary conditions of the maximum principle have the form

$$\dot{\boldsymbol{\lambda}}_1 = -\frac{\partial H_{np}}{\partial \mathbf{x}_1}, \quad \mathbf{0} = -\frac{\partial H_{np}}{\partial \mathbf{x}_2}, \quad \dot{\boldsymbol{\psi}}_1 = -\frac{\partial H_{np}}{\partial \boldsymbol{\xi}_1}, \quad \dot{\boldsymbol{\psi}}_i = -\frac{\partial H_{np}}{\partial \boldsymbol{\xi}_i} = -\boldsymbol{\psi}_{i-1}, \quad i = 2, \ldots, s \tag{6.1}$$

$$H_{np\max} = \max_{v} H_{np} : \boldsymbol{\psi}_s^T \mathbf{v} \Rightarrow \max_{v} \tag{6.2}$$

Consequently, for non-proper DS, the optimization procedure must also involve higher-order time derivatives of the control inputs, in accordance with integrator chain (3.1), (3.2). In addition, the constraint on the control $\mathbf{u} \in U$ must be treated as a constraint $\xi_1 \in U$ in the extended phase space (3.4). In practice, it is difficult to realize the maximum condition (6.2), because the constraint on $\mathbf{v}$ is not known in advance. One must therefore consider the appropriate formulation of the optimization problem. Condition (6.2) gives a hint that the problem may not have been correctly posed from the start. In many applications, the substance of the problem is sometimes such that a constraint $\mathbf{v} \in V$ makes more sense than $\mathbf{u} \in U$. If the condition $\mathbf{u} \in U$ is replaced by $\mathbf{v} \in V$, then conditions (6.1) and (6.2) lead to an ordinary optimization procedure. Then, in order to obtain an optimal (fictitious) control $\mathbf{v}(t)$, one must consider a two-point boundary-value problem. In accordance with integrator chain (3.1), (3.2), this control will not be static (proportional) but dynamic.

For the control mechanism of Section 2, one needs an integrator chain

$$\boldsymbol{\xi}_1 = \mathbf{u}, \quad \boldsymbol{\xi}_2 = \dot{\mathbf{u}}, \quad \mathbf{v} = \ddot{\mathbf{u}} : \quad \dot{\boldsymbol{\xi}}_1 = \boldsymbol{\xi}_2, \quad \dot{\boldsymbol{\xi}}_2 = \mathbf{v} \tag{6.3}$$

Consequently, the maximum principle must be applied to system (1.1), extended by the addition of Eqs (2.5) and (6.3). The total dimensionality of the extended system is $n + s = 5 + 2 = 7$. One then has to solve the conjugate equations (6.1) and maximum condition (6.2). Unconstrained optimization problems with a quadratic performance criterion were considered in [5].

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